1 Introduction

One of the outstanding problems in the analysis of mechanical properties of fiber composites is the prediction of failure due to a specified state of stress. The microstructural aspects of failure are of such complexity that there is little hope of resolution of this problem on the basis of micromechanics methods. Such methods would require analytical detection of successive microfailures in terms of microstress analysis and microfailure criteria and prediction of the coalescence of some of them to form macrofailures which is an intractable task.

A more hopeful approach is to assume the existence of a three-dimensional failure criterion in terms of macrovariables, such as average stresses or strains. A convenient mathematical representation of such criteria is in terms of polynomials. It is then necessary to determine the coefficients of the polynomial in terms of test results which can be conveniently obtained in the laboratory, such as uniaxial tension or compression, pure shear, etc.

The problem in this sense is superficially reminiscent of that of the construction of an initial yield criterion for an elasto-plastic material. Indeed, in one of the first contributions to the subject Tsai [1] assumed that the failure criterion of a unidirectional fiber composite has the same mathematical form as the yield criterion of an orthotropic ideally plastic material as given by Hill [2] in the form

\[ A(\sigma_{11} - \sigma_{22})^2 + B(\sigma_{22} - \sigma_{33})^2 + C(\sigma_{33} - \sigma_{11})^2 + 2D\sigma_{12}^2 + 2E\sigma_{23}^2 + 2F\sigma_{31}^2 = 1 \] (1)

The advantage of the form (1) is that it permits easy determination of the coefficients in terms of one-dimensional uniaxial and shear failure stresses. But its disadvantage is that it is based on the assumption that isotropic stress has no effect. It is for that reason that failure stresses. But its disadvantage is that it is based on the assumption of one-dimensional uniaxial and shear failure stresses. But its disadvantage is that it is based on the assumption that isotropic stress has no effect.

Failure Criteria for Unidirectional Fiber Composites

Three-dimensional failure criteria of unidirectional fiber composites are established in terms of quadratic stress polynomials which are expressed in terms of the transversely isotropic invariants of the applied average stress state. Four distinct failure modes—tensile and compressive fiber and matrix modes—are modeled separately, resulting in a piecewise smooth failure surface.

While such an assumption may be a good approximation for initial yielding of a metal, it is certainly not valid for isotropic tension of a fiber composite. Hoffman [3] has modified (1) by adding linear stress terms to it so as to account for unequal failure stresses in tension and compression with a single quadratic expression.

Tsai and Wu [4] represented the failure criterion as a general quadratic in the stresses, thereby eliminating the special dependence on the normal stresses of (1). In their notation

\[ F_{ij}\sigma_{ij} + F_i\sigma_i = 1 \] (2)

where

\[ i = 1, 2, 3, 4, 5, 6 \]

\[ 1 = 11, 2 = 22, 3 = 33, 4 = 31, 5 = 23, 6 = 12. \]

For an orthotropic material with unequal strengths in tension and compression

\[ F_{11} = \frac{1}{N_1 N_1^{-1}} \] (3a)

\[ F_1 = \frac{1}{N_1^{-1} - \frac{1}{N_1^{-1}}} \] (3b)

\[ F_{66} = \frac{1}{S_6^2} \] (3c)

where \( N_1, N_1^{-1} \) are tensile and compressive strengths, respectively, in 1-direction and \( S_6 \) is shear strength in the 12 plane. The coefficients \( F_{22}, F_{33}, F_{34}, F_{55}, F_{23}, F_{53} \), are given by analogous expressions.

Since the failure of the material is insensitive to a change of sign of shear stress (in contrast to a change of sign of normal stress) all terms containing a shear stress to first power must vanish. Therefore, the only surviving unknown coefficients are \( F_{22}, F_{23,} F_{34}, F_{55}, F_{23}, F_{53} \). Tsai and Wu propose to determine these coefficients by biaxial failure tests. Unfortunately, such tests are complicated and expensive.

The criterion (2) is a definite improvement over previous criteria because of its generality and versatility and has also provided good fit with test data. Nevertheless, its utilization and interpretation raise some intrinsic problems. Underlying these is the fact that a fiber composite consists of mechanically very dissimilar phases: stiff elastic brittle fibers and a compliant yielding matrix. Consequently, the
failure occurs in very different modes. Thus the fibers may rupture in tension or buckle in compression, or the matrix may fail due to loads transverse to the fibers. It is not evident that all of the distinct failure modes can be represented by a single smooth function such as (2).

An hexagonal crystal has macroscopically the same symmetry as a transversely isotropic fiber composite, yet the failure modes of these materials are entirely different. It would hardly be possible to bring forth these differences by mathematically similar smooth failure criteria which only differ in the numerical values of the coefficients.

To point out some difficulties specific to fiber composites let (2) be specialized to a unidirectionally reinforced lamina in plane stress. Then the failure criterion is

$$F_1 = o_{11} + 2o_{22} + o_{12} = 1 \quad (4)$$

where $x_1$ is fiber direction and $x_2$ transverse to it.

All coefficients except $F_{12}$ are determined by expressions of type (3). To determine $F_{12}$ one can use a variety of biaxial failure tests: tension-tension, compression-compression, tension-compression. It is not evident that the values of $F_{12}$ determined by these different tests will be the same or even close.

Suppose $F_{12}$ has been determined and it is desired to utilize (4) to predict failure for the biaxial tensile state of stress:

$$\sigma_{11} = \sigma \quad \sigma_{22} = \alpha \sigma \quad \sigma_{12} = 0$$

where $\alpha$ is a given number and $\sigma$ is to be determined. Insertion of this state of stress into (4) will result in a quadratic equation for $\sigma$. Because of the forms (3a,b) the coefficients of the quadratic depend on the tensile and compressive failure stresses $N_1^t, N_1^c, N_2^t, N_2^c$. It follows that failure under biaxial tensile stress depends on the values of the compressive failure stresses and this is physically unacceptable.

The foregoing difficulties provide the motivation to represent the failure criterion of a unidirectional fiber composite in piecewise smooth form, where each smooth branch represents one distinct failure mode. Such an approach has been introduced in [5] in the context of static and fatigue failure of fiber composites in plane stress.

2 Development of Failure Criteria

The failure criteria just discussed, as well as the ones to be established, are quadratic in the stresses. It should be emphasized that the choice of quadratics is based on curve fitting considerations and not on physical reasoning. The actual failure criterion is some closed surface in stress space, when all failure stresses are finite. An infinite failure stress may be used as a convenient idealization to indicate an order of magnitude difference, e.g., failure under isotropic pressure.

The simplest approximation of a failure surface is by planes parallel to coordinate planes, e.g., rectangular parallelepiped in three-dimensional stress space. Such representations have their uses for constant stress criteria in fixed directions with respect to material axes, but are in general of insufficient accuracy due to neglect of stress interaction effects.

The next approximation would consist of oblique planes which intersect the stress axes at the appropriate one-dimensional ultimate stresses. This may be termed the linear approximation and our experience shows that it underestimates the strength of the material and is therefore inscribed within the actual failure surface. However, linear approximations are very useful to model failure criteria of rocks [6].

The next order of approximation is quadratic, Fig. 1. It is the simplest presentation which can fit the data reasonably well, and in view of the significant scatter of failure test data it hardly seems worthwhile to employ cubic or higher approximations. It is unfortunate that the quadratic nature of stress-energy density forms has at times led to "physical" interpretation of quadratic failure criteria (or of quadratic initial yielding criteria in plasticity).

Literally all unidirectional fiber composites are transversely isotropic with respect to fiber direction, since the fibers are randomly placed. Consider such a fiber-reinforced cylindrical specimen which is referred to a system of axes $x_1$ in fiber direction and $x_2, x_3$ in the transverse directions. It follows from the transverse isotropy that the failure criterion must be invariant under any rotation of the $x_2, x_3$ axes around $x_1$. Therefore the failure criterion can be as a function of the stress invariants under such rotations. Such invariants are (see, e.g., [7]):

$$I_1 = o_{11} \quad (5a)$$

$$I_2 = o_{22} + o_{33} \quad (5b)$$

$$I_3 = (o_{12})^2 + (o_{23})^2 + (o_{31})^2 \quad (5c)$$

$$I_4 = (o_{12})^2 + (o_{23})^2 \quad (5d)$$

$$I_5 = 2(o_{12}o_{23} - o_{31})^2 \quad (5e)$$

A simple construction of these invariants is given in the Appendix to this paper. It is of interest to note that the second of (5c) is the square of the maximum transverse shear stress while (5d) is the square of maximum axial shear stress. The two alternative forms (5c) are related in terms of (5b). For present purposes it is more convenient to use the first of (5c).

In view of the choice of quadratic approximation (5e) cannot appear in the failure criterion. Thus the most general transversely isotropic quadratic approximation will have the form

$$A_1 I_1 + B_1 I_1^2 + A_2 I_2 + B_2 I_2^2 + C_1 I_3 I_1 + A_3 I_3 + A_4 I_4 = 1 \quad (6)$$

The one-dimensional failure stresses of the material are denoted as follows:

$$\sigma_A^t = \text{tensile failure stress in fiber direction}$$

$$\sigma_A^c = \text{compressive failure stress in fiber direction (absolute value)}$$

$$\sigma_T = \text{tensile failure stress transverse to fiber direction}$$

$$\tau_T = \text{transverse failure shear; } \sigma_{nt} \text{ in Fig. 2(b)}$$

$$\tau_A = \text{axial failure shear; } \sigma_{nt} \text{ in Fig. 2(b)}$$

It follows at once by application of (5) and (6) to pure transverse or axial shear:

$$A_3 = \frac{1}{\tau_T^2}$$

$$A_4 = \frac{1}{\tau_A^2} \quad (7)$$

Observation of failure in unidirectional fiber composites indicates
that there are two primary failure modes: a fiber mode in which the composite fails due to fiber rupture in tension or because of fiber buckling in compression and a matrix mode in which a plane crack parallel to the fibers occurs.

It may be argued that in the event that a failure plane can be identified, the failure is produced by the normal and shear stresses on that plane. For the fiber mode the failure plane is approximately the $x_3x_3$ plane. Therefore, the stresses producing this failure are $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{33}$ (Fig. 2(a)). A possible counter argument is that transversely isotropic compression $\sigma_{22} = \sigma_{33} = -\sigma$ would impede fiber buckling produced by compressive $\sigma_{11}$, thus requiring an interaction mechanism of these normal stresses. However, an analysis due to Rosen [8] indicates that fibers under axial compression buckle in a shear mode, thus independently of transverse normal stress. Accordingly such interaction will be assumed negligible, pending experimental investigation of this failure mode.

The matrix mode is a planar fracture in fiber direction, Fig. 2(b). The stresses on this plane are $\sigma_{nn}$, $\sigma_{nt}$, and $\sigma_{tt}$. The first two are expressed in terms of the stresses, $\sigma_{22}$, $\sigma_{33}$, and $\sigma_{23}$, while the last is expressed in terms of $\sigma_{12}$ and $\sigma_{13}$. Therefore, $\sigma_{11}$ does not enter into this failure mode.

If these conclusions are combined with the quadratic approximation (8) and the results (7) it follows that the failure criteria for these two modes are:

**Fiber Mode**

$$a^4\sigma_{11} + B_J^4\sigma_{11}^1 + \frac{1}{\tau_1^2}(\sigma_{12} + \sigma_{13}) = 1$$  \hspace{1cm} (8)

**Matrix Mode**

$$A_m(\sigma_{22} + \sigma_{33}) + B_m(\sigma_{22} + \sigma_{33})^2 + \frac{1}{\tau_n^2}(\sigma_{23} - \sigma_{22}\sigma_{33}) + 2\frac{1}{\tau_1^2}(\sigma_{12} + \sigma_{13}) = 1$$  \hspace{1cm} (9)

Failure mechanisms and failure stresses are quite different for tension and compression in fiber direction as well as for tension and compression transverse to fibers. Consequently each primary mode is further subdivided into tensile and compressive modes. These will now be discussed and modeled, separately.

**Tensile Fiber Mode** $\sigma_{11} > 0$. The simple uniaxial tensile test information, $\sigma_{11}^1 = \sigma_1^2$, provides only one equation for determination of the two coefficients $A_J$ and $B_J$.

Failure data for combined $\sigma_{11}$, $\sigma_{12}$ stressing (e.g., tension-torsion) are necessary to provide a necessary additional equation.

It is to be expected that tensile $\sigma_{11}$ and shear $\sigma_{12}$ are mutually weakening. If so, the failure locus for these two stresses must be convex. A possibly satisfactory simple approximation would be an ellipse with the axes. With this approximation (8) becomes

$$\frac{\sigma_{11}^2}{\sigma_1^2} + \frac{\sigma_{12}^1}{\tau_1^2} = 1$$  \hspace{1cm} (10)

A still more drastic approximation is to simply use the maximum stress criterion

$$\sigma_{11} = \sigma_1^2$$  \hspace{1cm} (11)

for tensile failure in fiber direction, thus disregarding the shear contribution.

**Compressive Fiber Mode** $\sigma_{11} < 0$. The only available information is that $\sigma_{11} = -\sigma_1^2$, so again there is insufficient information to find $A_J$ and $B_J$ in (8). It is not known to the writer whether axial shear stress produces a weakening or strengthening effect on compressive strength in fiber direction. Consequently the failure criterion is represented temporarily in the simple maximum stress form

$$\sigma_{11} = -\sigma_1^2$$  \hspace{1cm} (12)

The dependence of both fiber modes on the axial shear stresses does not appear well understood at the present time and deserves careful experimental investigation.

**Matrix Modes.** Matrix mode modeling is more complicated since the failure plane, Fig. 2(b), is not a priori identified. One possible procedure would be to formulate a surface failure criterion which would depend on $\sigma_{nn}$, $\sigma_{nt}$, and $\sigma_{tt}$. For example, a simple choice is

$$f(\sigma_{nn}, \sigma_{nt}, \sigma_{tt}) = \frac{\sigma_{nn}^2}{\tau_n^2} + \frac{\sigma_{nt}^2}{\tau_{nt}^2} + \frac{\sigma_{tt}^2}{\tau_t^2} = 1$$  \hspace{1cm} (13)

for the case of tensile transverse normal stress. A different failure criterion should be used for compressive $\sigma_{nn}$. If $\sigma_{nn}$, $\sigma_{nt}$, $\sigma_{tt}$, are expressed by tensor transformation in terms of the stresses $\sigma_{22}$, $\sigma_{33}$, $\sigma_{23}$, and the angle $(x_n, n) = \theta$, then (13) will be of the general form

$$g(\sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{12}, \sigma_{13}, \theta) = 1$$  \hspace{1cm} (14)

and failure will occur on the plane defined by $\theta$ which makes the left side of (14) a maximum. The failure criterion would then be given by (14) with $\theta = \theta_0$.

This procedure is somewhat reminiscent of Mohr failure theory as used in soil mechanics. While it may appear attractive because of its sound physical basis, it is unfortunately difficult to use. Even with the simple surface failure criterion (13) the extremum problem is quite complicated and definitely not quadratic. Furthermore, the failure criterion (13) is subject to the same difficulty encountered in the fiber modes in that presence of a linear term in $\sigma_{11}$ may be necessary. Similar difficulties are encountered for surface failure criterion for compressive $\sigma_{nn}$.

Finally, the extremum problem which must be solved is even more complicated than the one previously mentioned, since it must take into account the difference of failure criteria for tensile and compressive $\sigma_{nn}$. Consequently, the approach outlined will not be pursued at the present time.

Returning to the simple matrix mode criterion (9) it will be assumed that it represents a tensile mode defined by $\sigma_{nn} > 0$ and a compressive mode defined by $\sigma_{nn} < 0$. The coefficients $A_m$, $B_m$ for each mode are different and will be distinguished by $+$ and $-$ superscripts.

**Tensile Mode** $\sigma_{nn} > 0$. The available simple information is $\sigma_{22} = \sigma_2^2$ which, when used in (9) implies that

$$A_m^+\sigma_2^2 + B_m^+\sigma_2^2 = 1$$  \hspace{1cm} (15)

Using arguments similar to those leading to (10), $A_m^+$ is set equal to zero as an approximation. This results in the failure criterion

$$\frac{1}{\sigma_2^2}(\sigma_{22} + \sigma_{33})^2 + \frac{1}{\tau_n^2}(\sigma_{23} - \sigma_{22}\sigma_{33}) + \frac{1}{\tau_1^2}(\sigma_{12} + \sigma_{13}) = 1$$  \hspace{1cm} (16)

**Compressive Mode** $\sigma_{nn} < 0$. In this case the available simple information is $\sigma_{22} = -\sigma_2^2$, which when used in (9) implies that

$$-A_m^-\sigma_2^2 + B_m^-\sigma_2^2 = 1$$  \hspace{1cm} (17)

To obtain additional information it may be argued that if the material is failed in transversely isotropic pressure $\sigma_{22} = \sigma_{33} = -\sigma$, all others vanish, this pressure will be very much larger than the compressive uniaxial failure stress. Thus

$$\sigma = \sigma_2^2$$  \hspace{1cm} (18)

To utilize this condition $A_m^-$ and $B_m^-$ are first determined in terms of $\sigma_2^2$ and $\sigma$ and the condition (18) is then taken into account by retention of only first-order terms in $\sigma_2^2/\sigma$ in the expressions. The resulting failure criterion is

$$\frac{1}{\sigma_2^2}(\sigma_2^2)^2 - 1 = \frac{1}{4\tau_n^2}(\sigma_{22} + \sigma_{33}) + \frac{1}{\tau_n^2}(\sigma_{23} - \sigma_{22}\sigma_{33}) + \frac{1}{\tau_1^2}(\sigma_{12} + \sigma_{13}) = 1$$  \hspace{1cm} (19)

In order to determine which of the failure criteria (16), (19) should be used it is necessary to know the sign of $\sigma_{nn}$ on the failure plane.
Unfortunately, however, this requires prior identification of the failure plane in terms of the extremum problem previously discussed. Not only is such a procedure very complicated but, as has been pointed out in the foregoing, is incompatible with the quadratic approximation of failure criteria. Present development will be continued in terms of quadratic representations, but this will inevitably lead to certain requirements with regard to the failure plane location.

To simplify matters and without any loss of generality, the matrix mode failure criteria will be expressed in terms of the principal stresses \( \sigma_2, \sigma_3 \) associated with the plane stress system \( \sigma_{22}, \sigma_{33}, \sigma_{23} \) and the maximum axial shear stress \( \tau_n = \sqrt{\sigma_{12}^2 + \sigma_{13}^2} \). This permits representation of matrix mode failure criteria in three-dimensional stress space \( \sigma_2, \sigma_3, \tau_n \).

The normal stress \( \sigma_{nn} \) on a plane with orientation \( \theta \) relative to \( x^2 \)-axis is given by

\[
\sigma_{nn} = \sigma_2 \cos^2 \theta + \sigma_3 \sin^2 \theta
\]

Evidently

\[
\sigma_{nn} \geq 0 \quad \text{when} \quad \sigma_2, \sigma_3 \geq 0 \quad (20a)
\]

\[
\sigma_{nn} \leq 0 \quad \text{when} \quad \sigma_2, \sigma_3 \leq 0 \quad (20b)
\]

Thus (16) applies for the first quadrant and (19) for the third quadrant of the \( \sigma_2, \sigma_3 \)-plane, but in the second and fourth quadrants \( \sigma_{nn} \) can have either sign.

Let (16), (19) be represented in the general form (9) in present stress variables. Thus

\[
A_\sigma^\prime (\sigma_2 + \sigma_3) + B_\sigma^\prime (\sigma_2 + \sigma_3)^2 - \frac{\sigma_2 \sigma_3}{\tau_n^2} + \frac{\tau_n^2}{\tau_n^2} = 1 \quad (21a)
\]

\[
A_\tau^\prime (\sigma_2 + \sigma_3) + B_\tau^\prime (\sigma_2 + \sigma_3)^2 - \frac{\sigma_2 \sigma_3}{\tau_n^2} + \frac{\tau_n^2}{\tau_n^2} = 1 \quad (21b)
\]

The trace of the intersection of (21a, b) in the \( \sigma_2, \sigma_3 \)-plane is given by either of the two straight lines

\[
\sigma_2 + \sigma_3 = 0 \quad (22a)
\]

\[
\sigma_2 + \sigma_3 = -\frac{A_\sigma^\prime - A_\tau^\prime}{B_\sigma^\prime - B_\tau^\prime} \quad (22b)
\]

The criterion (16) must apply above the straight line and the criterion (19) below it. If (22b) is the intersection trace then it must intersect either the first or the third quadrants. In the first case it would follow that (21b) applies to part of the first quadrant which contradicts (20b). The conclusion is that only (22a) is compatible with (20) and is therefore the only acceptable intersection trace. Therefore

\[
\sigma_{nn} \geq 0 \quad \text{when} \quad \sigma_2 + \sigma_3 \geq 0 \quad (23a)
\]

\[
\sigma_{nn} \leq 0 \quad \text{when} \quad \sigma_2 + \sigma_3 \leq 0 \quad (23b)
\]

In view of (19) it is not difficult to see that the inequalities (21) and (23) are compatible only when \( \theta = \pm 45^\circ \). (This is easily perceived by consideration of the regions of validity of the inequalities in the \( \sigma_2, \sigma_3 \)-plane.) It follows that the quadratic representation implies that failure always occurs on the maximum transverse shear plane. This is difficult to accept as a general conclusion. A counterexample is a state of stress dominated by axial shear in which case the failure plane or orientation would more likely be governed by the direction of the maximum axial shear.

However, this difficulty is a direct consequence of the quadratic approximation. Higher-order approximations contain more disposable coefficients, are therefore more flexible and can better accommodate necessary physical ingredients but they would also require complicated and expensive combined stress testing in order to determine the increased number of coefficients.

The quadratic failure criteria derived are now summarized:

**Tensile Fiber Mode** \( \sigma_{11} > 0 \)

\[
\frac{\sigma_{11}}{\sigma_A^2} + \frac{1}{\tau_n^2} (\sigma_{12}^2 + \sigma_{13}^2) = 1 \quad (10)
\]

or

\[
\sigma_{11} = \sigma_A \quad (11)
\]

**Compressive Fiber Mode** \( \sigma_{11} < 0 \)

\[
\sigma_{11} = -\sigma_A \quad (12)
\]

**Tensile Matrix Mode** \( \sigma_{22} + \sigma_{33} > 0 \)

\[
\frac{1}{\sigma_1^2} (\sigma_{22} + \sigma_{33})^2 + \frac{1}{\tau_n^2} (\sigma_{22} - \sigma_{23}) + \frac{1}{\tau_n^2} (\sigma_{12}^2 + \sigma_{13}^2) = 1 \quad (13)
\]

**Compressive Matrix Mode** \( \sigma_{22} + \sigma_{33} < 0 \)

\[
\frac{1}{\sigma_1^2} (\sigma_{22} + \sigma_{33})^2 + \frac{1}{\tau_n^2} (\sigma_{22} + \sigma_{23})^2 + \frac{1}{\tau_n^2} (\sigma_{22} - \sigma_{23}) + \frac{1}{\tau_n^2} (\sigma_{12}^2 + \sigma_{13}^2) = 1 \quad (19)
\]

A schematic drawing of matrix failure criteria in the four quadrants of transverse plane principal stress is shown in Fig. 3. It should be noted that the ultimate transverse shear stress, \( \tau_n \), which appears in the failure criteria is notoriously difficult to measure. However, no general failure criterion could avoid inclusion of this quantity.

3 Plane Stress Failure Criteria

The case of plane stress is of considerable practical significance since such is the state of stress in the laminae of a laminate at sufficient distance from the edges. Let the nonvanishing stress components be \( \sigma_{11} \) in fiber direction, \( \sigma_{22} \) transverse to fibers and \( \sigma_{12} \) in plane or axial shear. Then it follows from the three-dimensional failure criteria that failure modes and their analytical descriptions are:

**Tensile Fiber Mode**

\[
\left( \frac{\sigma_{11}}{\sigma_A^2} \right)^2 + \left( \frac{\sigma_{12}}{\tau_A^2} \right)^2 = 1 \quad \sigma_{11} > 0 \quad (24)
\]

**Fiber Compressive Mode**

\[
\sigma_{11} = -\sigma_A \quad \sigma_{11} < 0 \quad (25)
\]

**Tensile Matrix Mode** \( \sigma_{22} > 0 \)

\[
\left( \frac{\sigma_{22}}{\sigma_A^2} \right)^2 + \left( \frac{\sigma_{12}}{\tau_A^2} \right)^2 = 1 \quad (26)
\]
The case of tensile modes (24), (26) can be conveniently tested by means of an off-axis specimen, Fig. 4, under uniaxial tension. The state of stress with respect to the material axes is then

\[ \sigma_{i1} = \sigma \cos^2 \theta \]
\[ \sigma_{22} = \sigma \sin^2 \theta \]
\[ \sigma_{12} = \sigma \sin \theta \cos \theta \]

(28)

Insertion of (28) into (24) and (26) predicts the off-axis failure stresses

\[ \sigma^2_{f,0} = \frac{1}{\cos^2 \theta [\cos^2 \theta / \sigma^2_T + \sin^2 \theta / \tau^2_T]} \]
\[ \sigma^2_{m,0} = \frac{1}{\sin^2 \theta [\sin^2 \theta / \sigma^2_T + \cos^2 \theta / \tau^2_T]} \]

(29)

where subscripts \( i \) and \( m \) indicate fiber and matrix modes, respectively.

Many off-axis test results have been reported in the literature, but in numerous cases coefficients of a polynomial which is supposed to represent a failure criterion have been obtained by best fit with the data. Such a procedure does not provide critical examination of a failure criterion.

In order to test a failure criterion it is necessary to examine prediction of failure under combined states of stress in terms of failure under single stress components. It also must be remembered that the significant scatter of test results tends to obscure verification. In this respect it should be noted that it is generally tacitly expected that a failure criterion will predict the mean of combined stress failure results in terms of the mean of one-dimensional failure stress results. There is in principle no statistical reason for such an assumption, but this subject is beyond the scope of the present discussion.

Pipes and Cole [8] reported off-axis tensile failure results for unidirectional boron-epoxy. In attempting to fit their results to the plane stress Tsai-Wu criterion (4) they found that \( F_{12} \) assumed considerably different values for the fiber orientations used (15°, 30°, 45°, 60°), the ratio of max \( F_{12} \) to min \( F_{12} \) being -51. They obtained good fit of (4) with experimental data by empirically setting \( F_{12} = 0 \).

The question of \( F_{12} \) significance has recently been considered in [10]. It was concluded on the basis of numerical computations that setting \( F_{12} = 0 \) in (4) or using the Hoffman criterion [3] would be practically acceptable as the expected errors would not exceed 10 percent. However, until physical arguments are supplied why \( F_{12} \) should vanish in a criterion such as (4) which contains a priori all terms on an equal rights basis, this must remain an empirical finding.

It should be pointed out that \( F_{12} \) is also absent from the present failure criterion, but this is due to consideration of the two different failure modes: fiber mode which is independent of \( \sigma_{22} \), and matrix mode which is independent of \( \sigma_{11} \).

Fig. 5 shows the experimental results of [9] and the failure criteria (24), (26) in terms of one-dimensional failure stresses reported in [9], showing reasonable agreement. It should be noted in this respect that these failure criteria are based on (8), (9) with vanishing \( A_f \) and \( A_m \). No doubt better fit with data could be obtained by retaining such coefficients in (24), (26) and obtaining them from fit to the data; but it is believed that the price paid in complication is hardly worth the return in accuracy.

Fig. 6 shows comparison with the failure criteria of experimental data for Glass/Epoxy off-axis specimens as reported in [5], demonstrating very good agreement. It should be noted in this respect that...
in [5] the fiber mode was modeled by the simple maximum stress criterion \( \sigma_1 = \sigma_2 \). This of course does not affect the form of the matrix mode, but is of importance for the value of the reinforcement angle, \( \theta_{im} \), which separates the two modes. With the present criteria this separation angle is about \( 8^\circ \), while with maximum stress in fiber direction the angle is \( 1^\circ 45' \). Test data for small angles \( \theta \) are needed in order to establish the relative accuracy of the two different fiber mode descriptions.

5 Conclusion

The principal theme of the present work is that a general failure criterion for unidirectional fiber composites should distinguish among the various different failure modes of the composite and model each of the modes separately. Consequently, the general failure criterion cannot be smooth but is piecewise smooth, consisting of smooth branches each of which models a distinct failure mode.

Such a failure criterion is physically more realistic than a completely smooth criterion since it avoids prediction of multiaxial tensile (compressive) modes—such a simple criterion was not sufficient to determine the coefficients in unambiguous fashion. A most important task is to provide carefully controlled test data for accurate modeling of failure modes. Such tests should also clarify the question of the validity of the quadratic approximation. As has been pointed out, quadratic failure criteria imply that the matrix mode failure plane is the maximum transverse shear plane and this does not seem acceptable in a general sense. It can only be hoped that the quadratic approximation will prove sufficiently accurate in spite of this physical shortcoming, for the alternative of higher-order failure criteria is very complicated.

Finally, the statistical scatter aspects of the failure must be recognized and incorporated by a theory which would predict statistical parameters of failure under combined stress terms of known statistics of failure under simple stress.

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It is a pleasure to acknowledge helpful discussions with B. Walter Rosen.

References


APPENDIX

Transversely Isotropic Invariants of Second Rank Symmetric Tensor

The usual isotropic invariants of the stress tensor or any other second rank symmetric tensor are

\[
I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \quad (30a)
\]

\[
I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 \quad (30b)
\]

\[
I_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{13}^2\sigma_{21} - \sigma_{23}^2\sigma_{12} - \sigma_{31}^2\sigma_{23} \quad (30c)
\]

Expressions (1) are invariants for the set of all rotations of a coordinate system. Consider the subset of rotations in which the \( x_1 \)-axis remains fixed. Evidently (1) are invariants also for such rotations.

A most elementary calculation shows that (5a) and (5d) are invariants for rotation around \( x_1 \)-axis. It follows from (30a) and (5a) that (5b) is an invariant. It then follows from (5a,b,d) and (30c) that (5c) is an invariant.

Finally, from (5a,c) and (30c) there follows the invariance of (5e).